Extreme natural risk and finance in ambiguous settings

A copula Approach

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Table of Contents

Table of Contents

Lecture's focus

The goal

The aim of this lecture is to focus on:

- the modelling of extreme natural events which could leads to instability in financial sector
- the modelling of the financial instruments related in some way to the natural events, which involves ambiguity feature at different levels.

To do so we consider a multidimensional framework:

- a financial network of banks that trade external assets and CAT bonds;
- CAT bonds act as insurance on a single natural event, no reinsurance;
- mispricing on CAT bonds, rising from uncertainty on the probability of occurrence of extreme climate-related disasters;
- correlation between shocks on CAT bonds and shocks on ext. assets.

Table of Contents

Preliminaries

The key result upon which all of this theory is built is the theorem of *probability integral transformation* proving that if one takes a random variable X with continuous distribution $F_X(x), \forall x \in \mathbb{R}$, the variable $u \equiv F_X(x), \forall x \in D_X$ that is the domain of X, has uniform distribution in the unit interval. $F^{[-1]} : (0,1) \to \mathbb{R};\ F^{[-1]}(u) = \inf\{t \in \mathbb{R} : F(t) \geq u\}$

Example

In the n-dimensional case, given $X_i, i = 1, ..., n$ with continuous distributions $F_{X_i}(x_i), x_i \in \mathbb{R}, i = 1, ..., n$, whose generalized inverse functions are $F_{X_i}^{[-1]}(u_i) = x_i, u_i \in [0, 1], x_i \in \mathbb{R}, i = 1, ..., n$, we rewrite the joint distribution in terms of uniformly distributed variables as

$$F_{X_1,\dots,X_n}(x_1, .., x_n) = F(F_{X_1}^{-1}(u_1), ..., F_{X_n}^{-1}(u_n)) \equiv C(\mathbf{u}),$$

where $u_i = F_{X_i}(x_i)$. Uniform variables are said *marginal distributions* while the function that links them is called *copula function*.

The requirements that have to be met by the function $C(\mathbf{u})$ in order to deliver a well-defined joint distribution are summarized in the following definition.

Definition

A copula is a function $C : [0,1]^n \to [0,1]$ satisfying the following requirements

1. *Grounded*: $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0, \forall i \in \mathbb{N} \subset (1, n)$,
2. *copula marginal*: $C(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_n)$, $\forall i \in \mathbb{N}$ is a $(n-1)$-copula function,
3. *n-Increasing*: $V_C(\mathbf{S}) \geq 0, \forall \mathbf{S}$ where $\mathbf{S}$ is a n-dimensional box defined by any couple of vectors $[\mathbf{u}, \mathbf{v}]$.

The relationship between copula function and joint distributions is clearly explained in *Sklar's theorem*, which is the basic pillar from which the theory of copula functions has grown up. This way copulas arise as the natural tool for statistical dependence modeling through *Sklar's theorem*.

Theorem (Sklar, 1959)

Let $(X_1, ...X_n) \equiv F_{X_1,...,X_n}$, *where* $X_i \sim F_{X_i}, i = 1, ..., n$. *There exists a n-dimensional copula C such that*

$$F_{X_1,...,X_n}(x_1, .., x_n) = C(F_{X_1}(x_1), ..., F_{X_n}(x_n)),$$

where $x_i \in \mathbb{R}, i = 1, ..., n$. *Moreover if* $F_{X_i}, i = 1, ..., n$ *are continuous, then C is unique. Conversely, if C is a n-dimensional copula and* $F_{X_i}, i = 1, ..., n$ *are distribution functions, then* $F_{X_1,...,X_n}$ *is a n-dimensional distribution function.*

Preliminaries

Sklar's theorem hence highlights the link between multivariate distributions functions and their univariate marginals providing a representation where we can separate the specification of the joint distribution from that of marginal distributions.
We observe that if the joint distribution is absolutely continuous and the marginals are strictly increasing, we are able to write the **joint density in term of the marginal's densities and the density of the copula functions**, i.e.

$$f_{X_1,\dots,X_n}(x_1,..,x_n) = c(F_{X_1}(x_1),\dots,F_{X_n}(x_n))\prod_{i=1}^{n} f_{X_i}(x_i),$$

where $x_i \in \mathbb{R}, i = 1,\dots,n$ and the densities are noted in small letters.

Fréchet-Hoeffding bounds of a n-dimensional copula

Theorem

Every copula satisfies the following inequality:

$$C^{-}(\mathbf{u}) \leq C(\mathbf{u}) \leq C^{+}(\mathbf{u}), \forall \mathbf{u} \in [0,1]^n$$

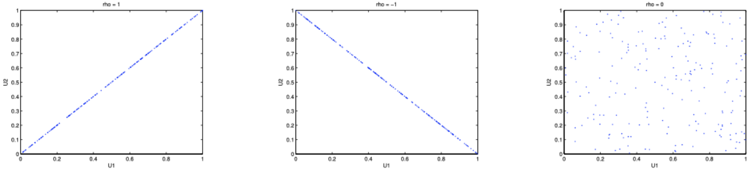
where the minimal bound $C^{-}(\mathbf{u})$ and the maximal one $C^{+}(\mathbf{u})$ are defined as

$$\begin{aligned} C^{-}(\mathbf{u}) &= \max\left(\sum_{i=1}^{n} u_i - n + 1, 0\right) \\ C^{+}(\mathbf{u}) &= \min(\mathbf{u}), \forall \mathbf{u} \in [0,1]^n. \end{aligned}$$

- The set $\{C^-, C^+\}$ defines the Fréchet-Hoeffding bounds of a n-dimensional copula: the upper Fréchet-Hoeffding bound which always satisfies the definition of copula function, is called *maximal copula* while the lower bound never satisfies the necessary conditions to be defined as a copula function for $n > 2$ (see Féron, 1956).
- It is important to observe that in the two-dimensional case, Fréchet-Hoeffding bounds represent perfectly negatively and positively dependent structures, respectively suggesting that the value of copula functions increase with the dependence.
- The case of no dependence at all is notably and it is represented by the *product copula* defined as

$$C^{\top}(\mathbf{u}) = \prod_{i=1}^{n} u_i.$$

Fréchet-Hoeffding bounds of a n-dimensional copula



(a) Minimal copula. (b) Maximal copula. (c) Product copula.

Figure 2.1: Scatter plots of 200 data simulated from the bivariate Fréchet-Hoeffding bounds copulas and the product copula.

Radial Symmetry and Exchangeability

Copulas can also be characterized with respect to different notions of symmetry. We consider in the following two main kind of symmetry.

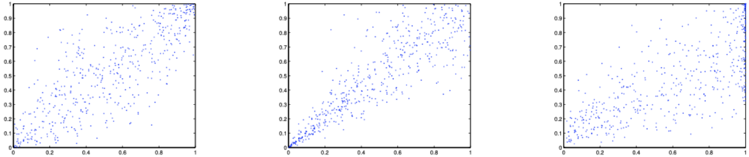
Definition (Radial symmetry)

A copula function C is radially symmetric if $C(\mathbf{u}) = C(\mathbf{1} - \mathbf{u})$, where $\mathbf{1}$ denotes the n-dimensional unit vector.

Definition (Exchangeability)

A copula C is exchangeable if $C(\mathbf{u}) = C(\mathbf{u}_\sigma)$, for any entries' permutation $\sigma : 1, ..., n \rightarrow 1, ..., n$.

Radial Symmetry and Exchangeability



(a) Radially symmetric and exchangeable copula.

(b) Radially asymmetric and exchangeable copula.

(c) Radially asymmetric and not exchangeable copula.

Figure 2.2: Scatter plots of 500 data simulated from a bivariate radially symmetric and exchangeable copula, from a radially asymmetric and exchangeable copula and from a radially asymmetric and not exchangeable copula.

Association Measures

To explain the dependence information concerning two random variables (X_1, X_2) in a single number, it is common to use association measures which are defined in term of their copula function C.

Definition (Kendall's τ and Spearman's ρ_S)

Kendall's τ is given by

$$\tau(C) = 4 \int_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1,$$

while Spearman's ρ_S by

$$\rho_S(C) = 12 \int_{[0,1]^2} C(u_1, u_2) du_1 du_2 - 3.$$

It holds true that $\tau(C), \rho_S(C) \in [-1, 1]$.

Tail Dependence

On the other hand to measure the strength of dependence in the tails of the joint distribution it is common to use the tail dependence coefficients, which are also purely copula-based measures.

Definition (Tail dependence coefficients)

The lower tail dependence coefficients is given by

$$\lambda_L(C) = \lim_{t \to 0} \frac{C(t,t)}{t}, = \lim_{t \to 0} P\left(X_2 \le F_2^{-1}(t) \,\middle|\, X_1 \le F_1^{-1}(t)\right)$$

while the upper tail dependence coefficient is given by

$$\lambda_U(C) = 2 - \lim_{t \to 1} \frac{1 - C(t,t)}{1 - t}. = \lim_{t \to 1} P\left(X_2 > F_2^{-1}(t) \,\middle|\, X_1 > F_1^{-1}(t)\right)$$

It holds true that $\lambda_L(C), \lambda_U(C) \in [0, 1]$.

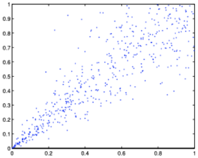
Tail Dependence and Rotation

A copula C is said to be lower/upper tail dependent if $\lambda_L(C) > 0/\lambda_U(C) > 0$. We observe that a radial symmetric copula C satisfies $\lambda_L(C) = \lambda_U(C)$. It is particularly useful to observe that the characterization of copula function based on the concepts of symmetry and tail dependence can be modified by a rotation of the copula itself. There are different kind of rotations among which we recall the **rotation by 180 degrees** defining the *survival copula* $\overline{C}$, i.e.

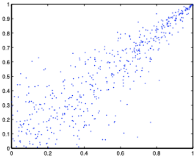
$$\overline{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2),$$

where $u_i \in [0, 1], i = 1, 2$. Clearly a radially symmetric copula C coincides with its survival version.

Tail Dependence and Rotation



(a) No rotation.



(b) Rotation by 180 degrees.

Figure 2.3: Scatter plots of 500 data simulated from a bivariate copula and its survival.

Kendall distribution function

Finally we observe that the copula function provides a mapping to $[0,1]$ whose distribution can be characterized using the notion of *Kendall function*.

Definition (Kendall function)

The Kendall function of a copula C is given by

$$K(z, C) = \mathbb{P}(C(\mathbf{u}) \leq z), z \in [0,1].$$

It holds that $z \leq K(z, C) \leq 1, z \in [0,1]$, whose bounds represent the case of perfect positive and negative dependence, respectively.

Kendall distribution function

Picture 1 gives an example of Kendall function showing its range delimited by the limiting bounds of the function itself.

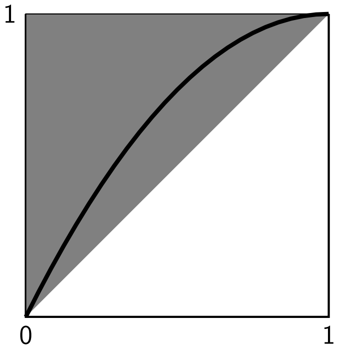


Figure: Graphical representation of the Kendall function. The gray area represents the limiting bounds of the function itself.

Copula Family: Elliptical copulas

The family of *elliptical copulas* are obtained directly from the class of elliptical distributions, recalling that by *Sklar's theorem* we have

$$C(\mathbf{u}) = F_{X_1,\dots,X_n}(F_{X_1}^{-1}(u_1),\dots,F_{X_n}^{-1}(u_n)),$$

where $u_i \in [0,1], i = 1,\dots,2$ and the joint distribution along with the marginals are elliptical distribution functions. The density of an elliptical copula C is then

$$c(\mathbf{u}) = \frac{f(F_{X_1}^{-1}(u_1),\dots,F_{X_n}^{-1}(u_n))}{\prod_{i=1}^{n} f_{X_i}(F_{X_I}^{-1}(u_i))},$$

where $u_i \in [0,1], i = 1,2$. For the Kendall function of elliptical copulas, however no close-form expression is available.
It holds true that all elliptical distributions are radially symmetric.

Gaussian copulas

Define $\Phi_n(\epsilon_1, ..., \epsilon_n; \mathbf{R})$ the n-dimension multivariate Gaussian distribution, linking together a set of n standardized variables with correlation represented by the n-dimension matrix $\mathbf{R}$. It is well known that the marginal distribution of each variable is also standard normal. Denote $\Phi(x)$ the univariate standard normal distribution. You can now rewrite the joint normal distribution as

$$C(\mathbf{u}) \equiv \Phi_n \left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n); \mathbf{R}\right). \tag{1}$$

This is known as *Gaussian copula* function, and it is by far the most widely used in financial applications.

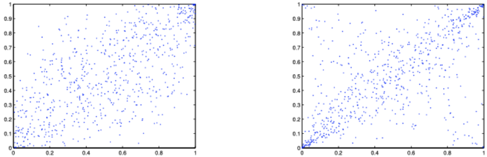
Gaussian copulas

The Gaussian copula is defined by $n(n-1)/2$ parameters, unless a specific structure of the correlation matrix is assumed. In the case of an exchangeable correlation matrix with $\rho_{jk} = \rho \in (-1/(n-1), 1)$ for all $j, k = 1, ..., n, j \neq k$, the Gaussian copula is exchangeable itself and is specified by only one parameter.
In the bivariate case the Gaussian copula depends only on the off-diagonal parameter of the correlation matrix, i.e. $\rho = \rho_{12}$. In this case both the Kendall's τ and Spearman's ρ_S can be represented in term of this parameter as

$$\begin{aligned} \tau(\rho) &= \frac{2}{\pi} \arcsin(\rho) \\ \rho_S(\rho) &= \frac{6}{\pi} \arcsin\left(\frac{\rho}{2}\right). \end{aligned}$$

Finally we observe that the Gaussian copula is tail independent, i.e. $\lambda_L(\rho) = \lambda_U(\rho) = 0$.

Elliptical copulas



(a) Gaussian copula. (b) Student's t copula.

Figure 2.5: Scatter plots of 500 data simulated from the bivariate Gaussian and t copula with $\rho = 0.7$ and $\nu = 1$.

Student's t copula

We know that in univariate models a way to introduce fat tails is to substitute the normal distribution with the so called *Student's t* distribution, in which a parameter ν, representing the *degrees of freedom*, is responsible for generating fat tails. The same can be done with copulas. So, define $\mathbf{T}_n(\epsilon_1, ..., \epsilon_n; \mathbf{R}, \nu)$ the joint *Student's t* distribution, and denote $\mathbf{T}_\nu(x)$ the univariate *Student's t* distributions. The *Student's t copula* is then defined as

$$C(\mathbf{u}) \equiv \mathbf{T}_n \left(\mathbf{T}_\nu^{-1}(u_1), ..., \mathbf{T}_\nu^{-1}(u_n); \mathbf{R}, \nu \right). \tag{2}$$

The Student's t copula is defined by $n(n-1)/2$ correlation parameters and additionally the degree of freedom parameter, that is $n(n-1)/2+1$ parameters in total. In the case of an exchangeable correlation matrix with $\rho_{jk} = \rho \in (-1/(n-1), 1)$ for all $j, k = 1, ..., n, j \neq k$, the Student's t copula is exchangeable itself and is specified by only two parameters. In the bivariate case the copula depends only on the off-diagonal parameter of the correlation matrix, i.e. $\rho = \rho_{12}$ and ν. In this case both the Kendall's τ can be represented in term of these parameters as

$$\tau(\rho) = \frac{2}{\pi} \arcsin(\rho),$$

while Spearman's ρ_S has no-closed form expression.
Like the bivariate Gaussian copula, it is symmetric, but differently from that, it has tail dependence, given by

$$\lambda_L(\rho,\nu) = \lambda_U(\rho,\nu) = 2T_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right).$$

Archimedean Copulas

Assume a function $\phi(x)$, called the generator of the Archimedean copula function. Now we use it to define the class of *Archimedean copulas* as follows

$$C(\mathbf{u}) \equiv \phi^{-1}\left(\sum_{i=1}^{n}\phi(u_i)\right).$$

To provide a general way to define the requirements that must be met to generate an Archimedean copula function, let us focus on the inverse of the generator $\psi(\cdot) \equiv \phi^{-1}(\cdot)$. In order to allow for every dimension n, this function is required to be infinitely differentiable, with derivatives of ascending order of alternate sign, and such that $\psi(0) = 1$ and $\lim_{x\to+\infty}\psi(x) = 0$.

Hence given a n-dimensional Archimedean copula that is absolutely continuous, i.e. $(\phi^{-1})^{n-1}$ exists and is absolutely continuous in $(0, \infty)$, it has a density given by

$$c(\mathbf{u}) = (\phi^{-1})^n(\phi(u_1) + ... + \phi(u_n)) \prod_{i=1}^{n} \phi'(u_i),$$

where $u_i \in [0, 1], i = 1, 2$.
The Kendall function of a n-dimensional Archimedean copula is very complex and is given in term of the generator and the highest order of derivatives of its inverse (see Barbe et al., 1996). In the bivariate case we have

$$K(z, \phi) = z - \frac{\phi(z)}{\phi'(z)},$$

where $z \in [0, 1]$.

Similarly the Kendall's τ can be written as

$$\tau(\phi) = 1 + 4 \int_0^1 \frac{\phi(z)}{\phi'(z)} \, dz,$$

while there is not a closed-form expression for the Spearman's ρ_S.
The tail dependence coefficients can be represented as follows

$$\begin{aligned}
\lambda_L(\phi) &= \lim_{t \to \infty} \frac{\phi^{-1}(2t)}{\phi^{-1}(t)} \\
\lambda_U(\phi) &= 2 - \lim_{t \to 0} \frac{1 - \phi^{-1}(2t)}{1 - \phi^{-1}(t)}.
\end{aligned}$$

Clayton copula

For example, $\psi(t) \equiv (1+\theta t)^{-1/\theta}$ that is the Laplace transform of the *Gamma* distribution, yields

$$\phi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right),$$

and the corresponding copula is most famous among Archimedean copulas, that is the *Clayton copula*

$$C(\mathbf{u}) = \max\left(\left(\sum_{i=1}^{n} u_i^{-\theta} - n + 1\right)^{-1/\theta}, 0\right).$$

The limiting cases of Clayton copula are independence if $\theta \to 0$, comonotonicity if $\theta \to \infty$ and bivariate countermonotonicity if $\theta \to -1$.

The corresponding Kendall's τ is given by

$$\tau(\theta) = \frac{\theta}{\theta + 2}.$$

Finally we point out that Clayton copula is a radially asymmetric copula and thus tail asymmetric, showing lower tail dependence for $\theta > 0$. In fact the tail coefficients are given by

$$\begin{aligned} \lambda_L(\theta) &= 2^{-1/\theta} \\ \lambda_U(\theta) &= 0. \end{aligned}$$

Frank copula

Other famous example in the class of Archimedean copulas is represented by the *Frank copula*, obtained from the generator

$$\phi(t) = -\ln\left(\frac{\exp(-\theta t) - 1}{\exp(-t) - 1}\right).$$

The *Frank copula* is then defined as

$$C(\mathbf{u}) = -\ln\left(1 + \frac{\prod_{i=1}^{n}(\exp(-\theta u_i) - 1)}{\exp(-\theta) - 1}\right).$$

It converges to independence for $\theta \to 0$ and to comonotonicity for $\theta \to \infty$, while in the bivariate case if $\theta \to -\infty$ covers the countermonotonicity.

The Kendall's τ and the Spearman's ρ_S can be written as

$$\begin{aligned}\tau(\theta) &= 1+\frac{4}{\theta}(D_1(\theta)-1)\\ \rho_S(\theta) &= 1-\frac{12}{\theta}(D_1(\theta)-D_2(\theta)),\end{aligned}$$

where $D_k(x)$ stands for the Debye function, i.e.

$$D_k(x) = \ frackx^k \int_0^x \frac{t^k}{e^t-1}\,dt,$$

where $x \in \mathbb{R}\setminus\{0\}, k \in \mathbb{N}$.
In the bivariate case the Frank copula is radially symmetric while it looses this property for $n > 2$, and it is not tail dependent, i.e. $\lambda_L(\theta) = \lambda_U(\theta) = 0$.

Gumbel copula

Another famous Archimedean copula is generated by $\phi(t) \equiv (-\ln t)^{\alpha}$ which yields the *Gumbel family*:

$$C(\mathbf{u}) = \exp\left(-\left(\sum_{i=1}^{n}(-\ln u_i)^{\theta}\right)^{1/\theta}\right),$$

whose limiting cases are independence for $\theta = 1$ and comonotonicity if $\theta \to \infty$. The Gumbel copula is limited to positive dependence only. In the jargon of copula functions, we say it is not *comprehensive*.

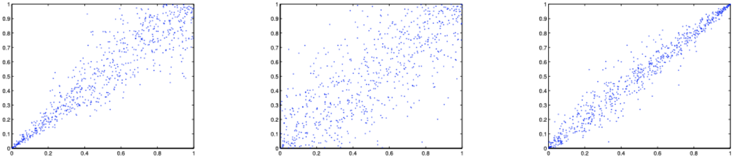
The Kendall's τ is given by

$$\tau(\theta) = 1 - \frac{1}{\theta},$$

while there is no closed-form expression of Spearman's ρ_S.
Gumbel copula has upper tail dependence. Finally we point out that Gumbel copula is a radially asymmetric copula and thus tail asymmetric, showing upper tail dependence given by

$$\begin{aligned} \lambda_L(\theta) &= 0 \\ \lambda_U(\theta) &= 2 - 2^{1/\theta}. \end{aligned}$$

Archimedean copulas



(a) Clayton copula. (b) Frank copula. (c) Gumbel copula.

Figure 2.6: Scatter plots of 500 data simulated from the bivariate Clayton, Frank and Gumbel copula with $\theta = 5$.

Joe copula

An other Archimedean copula which shows the same tail dependence coefficients as the Gumbel copula, is the *Joe family* whose generator is $\phi(t) \equiv -\log(1-(1-t)^{\theta})$. The corresponding copula is then:

$$C(\mathbf{u}) = 1 - \left(1 - \prod_{i=1}^{n}(1-(1-u_i)^{\theta})\right)^{1/\theta},$$

where $u_i \in [0,1], i = 1, ..., n$ and $\theta > 1$. The limiting cases are the independence for $\theta \to 1$ and the comonotonicity for $\theta \to \infty$.
We recall that it exists a closed-form representation of his Kendall's τ in term of the digamma function, which corresponds to the logarithmic derivative of the gamma function.

Extreme-Value copulas

Extreme-Value copulas, introduced by Pickands (1981), can be defined as the asymptotic limit of the component-wise maxima. Given d independent copies of a n-dimensional random vector $\mathbf{X}$ with copula C_0, we define $M_{d,i} = \max\{X_{1i}, ..., X_{di}\}$ that is the maximum of the ith component over d copies. Then the copula of $M_n = (M_{d,1}, ..., M_{d,n})$ is

$$C_{0(n)}(\mathbf{u}) = C_0(\mathbf{u}^{1/n})^n,$$

for all $u_i \in [0,1], i = 1, ..., n$. C is an extreme-value copula if there exists a copula C_0 which lies in the domain of attraction of C, i.e.

$$C_{0(n)}(\mathbf{u}) \rightarrow_{n\rightarrow+\infty} C(\mathbf{u}), \forall \mathbf{u} \in [0,1]^n.$$

Extreme-Value copulas

In the bivariate case an extreme-value copula is uniquely defined by the Pickands function $A : [0,1] \to [0.5,1]$ which is convex and such that $\max\{t, 1-t\} \leq A(t) \leq 1, \forall t \in [0,1]$. Hence the corresponding extreme-copula can be written as

$$C(u_1, u_2, A) = \exp\left(\log(u_1 u_2) A\left(\frac{\log u_2}{\log(u_1 u_2)}\right)\right), \quad (u_1, u_2) \in [0,1]^2.$$

On the other hand if C is a bivariate extreme-value copula, then there exists a Pickands function A which allows the above representation of the copula. The limiting case are represented by independence for $A(t) = 1, \forall t$ and the comonotonicity for $A(t) = \max\{t, 1-t\}$.

Finally we observe that an extreme-value copula is symmetric if and only if the Pickands function is symmetric about 0.5, i.e. $A(t) = A(1-t), \forall t \in [0,1]$. Moreover the bivariate extreme-value copula has an explicit Kendall function (see Ghoudi et al., 1998) which depends on the Pickands function, as follows

$$K(z, A) = z(1 + (\tau(A) - 1)\log z), \quad z \in [0,1],$$

where the Kendall's $\tau(A)$ of the extreme-value copula is defined as

$$\tau(A) = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t),$$

assuming that A' exists. Similarly the Spearmen's $\rho_S(A)$ can be represented in term of the Pickands function:

$$\rho_S(A) = 12 \int_0^1 \frac{1}{(1 + A(t))^2}\, dt - 3.$$

For that concerns the tail dependence, we can say that the extreme-value copulas are in general lower tail independent except the comonotonicity case, where $A(0.5) = 0.5$ and upper tail dependent but they are not radially symmetric except the limiting cases. In fact the tail dependence coefficients can be written as

$$\begin{aligned} \lambda_L(A) &= \begin{cases} 1 & \text{if } A(0.5) = 0.5 \\ 0 & \text{otherwise} \end{cases} \\ \lambda_U(A) &= 2(1 - A(0.5)). \end{aligned}$$

Genest and Rivest (1989) proved that the Gumbel copula is the only copula that is both Archimedean and an extreme-value copula characterized by the following symmetric about 0.5 Pickands function:

$$A(t) = (t^\theta + (1-t)^\theta)^{1/\theta}, \quad t \in [0, 1].$$

Archimedean copulas lie in the domain of attraction of Gumbel copula if there exists $\lim_{s\to 0} \frac{s\phi'(1-s)}{\phi(1-s)} \in [1,\infty]$ and then produce radially symmetric copulas.
An extension of Gumbel copula which allows for asymmetry is the **Tawn copula** whose Pickands function is given by

$$A(t,\theta,\psi_1,\psi_2) = (1-\psi_2)(1-t) + (1-\psi_1)t + ((\psi_2 t)^\theta + (\psi_1(1-t))^\theta)^{1/\theta},$$

where $\theta \geq 1$ and $\psi_1, \psi_2 \in [0,1]$. We observe that if $\psi_1 = \psi_2 = 1$ the Tawn copula reduces to the Gumbel copula while if $\psi_1 \neq \psi_2$ it shows an asymmetric behaviour. Moreover the independence case corresponds to the assumption $\theta = 1$ or $\psi_1 = 0$ or $\psi_2 = 0$ while for $\theta \to \infty$ it converges to the **Marshall-Olkin copula**.

Pure Hierarchical Copulas

Hierarchical Archimedean copulas have been introduced in Savu and Trede (2010) for Archimedean class to **encompass the major flaw of this class coming from the exchangeability**, i.e. the assumption that the dependence parameter is the same across the several pairs of assets. In fact, it is easy to verify that Archimedean copulas lead to the following result

$$C(1, ..., 1, u_i, 1, ..., 1, u_j, 1, ..., 1) = \phi^{-1}(\phi(u_i) + \phi(u_j)), \forall i, j$$

where ϕ is the generator of the Archimedean copula. A way to generalize this representation allowing for $C(u_i, u_j) \neq C(u_m, u_n), i \neq m, j \neq n$ is to impose a *hierarchical* representation.

Hierarchical Archimedean copulas can be **fully or partially nested** depending on the kind of aggregation we use. For example a *fully nested* copula in dimension $n = 3$ corresponds to

$$C(\mathbf{u}) = \phi_2^{-1}\left(\phi_2(u_3) + \phi_2 \circ \phi_1^{-1}(\phi_1(u_2) + \phi_1(u_1))\right)$$

where ϕ_1 represents the generator explaining the dependence between the first two variables, ϕ_2 stands for the highest level of dependence, i.e. between the previous couple and the third variable and $\circ$ is the composition operator. Note that in this 3-dimensional case, we have 2 nesting levels, meaning that a n-dimensional fully nested Archimedean copula is able to capture $n-1$ different pairwise dependence. Otherwise, C is called *partially nested* Archimedean copula.

Partially exchangeable copulas

It is based on **grouping aggregation**, which represents the second level of the hierarchy, and a final aggregation between groups, corresponding to the first level of the hierarchy. It is a particular partially nested copula where the **exchangeability between groups**, i.e. a partial exchangeability property, is preserved. In n-dimension and with pair aggregation, the resulting copula is $PE-HC(\mathbf{u}) \equiv$
$\phi_s^{-1}\left(\phi_s \circ \phi_1^{-1}(\phi_1(u_1)+\phi_1(u_2)) + ... + \phi_s \circ \phi_n^{-1}(\phi_n(u_{n-1})+\phi_n(u_n))\right)$,
where ϕ_s represents the generator of the Archimedean copula explaining the dependence **between couples**, $\phi_j, j=1,...,n$ represent the dependence **within couples** and $\circ$ is the composition operator. This representation uses different generators $\phi_1,...,\phi_n$ for the couples (u_1,u_2), (u_2,u_3) ,..., (u_{n-1},u_n) and a generator $\phi_s \circ \phi_i^{-1}, i=1,...,n$ to model the dependence of the pairs (u_1,u_3), (u_2,u_4), ...(u_{n-2},u_n).

For the generator $\phi_s \circ \phi_i^{-1}, i = 1, ..., n$ to be well defined, the **dependence parameter must be decreasing from the bottom to the highest hierarchical level** as proved in Embrechts at al. (2003).
So, denoting α_i the dependence parameter of the generator ϕ_i, in our case it must be $\alpha_s \leq \max\{\alpha_i, i = 1, ..., n\}$.
Moreover it is possible to choose generators of different Archimedean families. Note however that **some families are incompatible** with each other and will never form a valid copula. As for example Gumbel and Clayton generators are incompatible in hierarchical models (see Hofert, 2008).

Partially exchangeable and partially nested hierarchical aggregations.

Figure 2.7: Partially exchangeable hierarchical aggregations.

Not exchangeable copulas

This kind of hierarchy is recovered through a **nested aggregation** that proceeds by a first group aggregation and a series of single aggregations. In the following example we have a **fully nested copula** but we can imagine also a **partially nested case that proceeds by a nested aggregation by groups while by single ones**.

Not exchangeable copulas

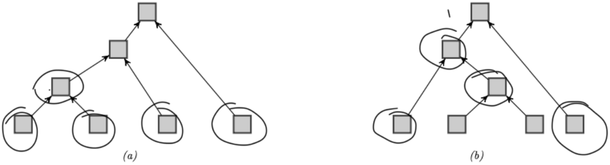


Figure 2.8: Not exchangeable hierarchical aggregations.

Not exchangeable copulas

In n-dimensions the resulting copula is

$$NE - HC(\mathbf{u}) \equiv \phi_{n-1}^{-1}\left(\phi_{n-1}(u_n) + \phi_{n-1} \circ \phi_{n-2}^{-1}\left(\phi_{n-2}(u_{n-1}) + ...+\right.\right.$$
$$\left.\left.+\phi_{n-2} \circ \phi_{n-3}^{-1}\left(\phi_{n-3}(u_{n-2}) + \phi_2 \circ \phi_1^{-1}(\phi_1(u_2) + \phi_1(u_1)\right.\right.\right.$$

For the nested copula to be well defined, we must assume the **dependence parameter is decreasing from ϕ_{n-1} to ϕ_1**, i.e. the highest level of the hierarchy must that with feeble dependence. Moreover also for the nested aggregation generating not exchangeable copulas, it would be possible to choose generators coming from different Archimedean families, if they are compatible according to the meaning specified by Hofert (2008).

Moreover we stress that in this kind of aggregation the way the classes are defined becomes relevant in order to recover the multivariate distribution. We can imagine to define several classes based on specified characteristics of the variables and through a clustering method. In this case we recover the **clusterized copulas**; here we need to aggregate groups of variables which can be assumed to be similar each other based on the similarity features used in the clustering phase. We disentangle two kind of *clusterized copulas*, i.e. the **homogeneous** class where the variables inside the same group are assumed to be homogeneous and the *not homogeneous* or **hierarchical** where we have a hierarchy also inside the groups. A special case of the homogeneous class where we assume to have only one level of dependence leads to an aggregation which is not hierarchical but which maybe useful in order to reduce the complexity of an high dimensional problem.

Clusterized homogeneous hierarchical copulas

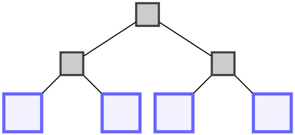
The class of *clusterized homogeneous copulas* (CHC for short in the following) had been introduced in Bernardi and Romagnoli (2013). They are grouping copulas whose classes, recovered through a clustering method, are assumed to be homogeneous and are related by a dependency possibly hierarchical. Therefore every class is characterized by a **perfect positive dependence on his centroid** while the aggregation **between classes maybe represented by more than one level of dependence**.

An important special copula of this class is the 1-level CHC copula.

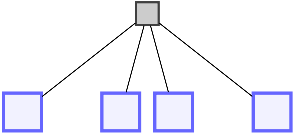
Definition

(1-CHC) A one level-clusterized homogeneous copulas is a grouping copula based on clustering method, with only one level of dependence and whose groups are assumed to be homogeneous.

Clusterized homogeneous hierarchical copulas



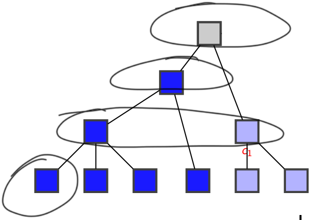
(a)



(b)

figureClusterized Homogeneous copulas: (a) 2-CHC and (b) 1-CHC.

Clusterized Hierarchical Copulas



figureClusterized Hierarchical copulas: three levels of dependence.

Compensation for the homogeneous feature

If we compensate with the dependence parameter, we have the following;

Definition

The Equivalent Dependence Parameter (EQDP) for the copula function C of group $i, i = 1, ..., n$ is α_i such that:

$$C(\mathbf{u}^i, \widehat{\alpha}_i) = C(\mathbf{c}_i, \alpha_i),$$

where $u_j^i, j = 1, ..., m$ are the heterogenous marginals of the group i, $C(\mathbf{c}_i, \alpha_i)$ is an m-variate copula with homogeneous marginals c_i equal to the centroid of the i-th cluster and $\widehat{\alpha}_i$ is the standard estimation of the dependence parameter within the heterogenous group for the same copula function.

Compensation for the homogeneous feature

On the other hand, if we decide to change the number of the variables to be considered into the homogenous group, we have the following;

Definition

The Equivalent Number of Variables (ENV) for the copula function C of group $i, i = 1, ..., n$ is the dimension m_i of copula C after the homogeneous approximation:

$$C(\mathbf{u}^i, \widehat{\alpha}_i) = C^{(m_i)}(\mathbf{c}_i, \widehat{\alpha}_i),$$

where $u^i_j, j = 1, ..., m$ are the heterogenous marginals of group i, $C^{(m_i)}(\mathbf{c}_i, \widehat{\alpha}_i)$ is an m_i-variate copula with homogeneous marginals c_i equal to the centroid of the i-th cluster and $\widehat{\alpha}_i$ is the standard estimation of the dependence parameter within the heterogenous group for the same copula function.

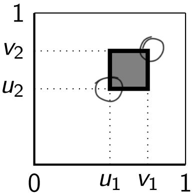
Volume of an n-dimensional copula

In a bivariate setting the volume of a copula function C between two points $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$, where $u_i \leq v_i, i = 1, 2$ is defined as

$$V_C(\mathbf{S}) = C(v_1, v_2) - C(u_1, v_2) - C(v_1, u_2) + C(u_1, u_2),$$

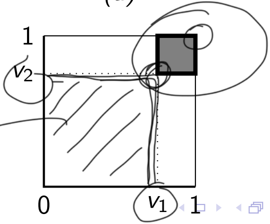
where $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$. This representation is easily understood if you look at the picture where the volume is geometrically represented.

Volume of an n-dimensional copula



(a)

$C(v_1, v_2)$



$\bar{C}$

An interesting interpretation of the volume concerns a particular box defined as $\mathbf{S} = [\mathbf{v}, \mathbf{1}]$ where $\mathbf{1}$ is a 2-dimensional unit vector. Through the same representation used before, we have

$$\begin{aligned} V_C(\mathbf{S}) &= C(1,1) - C(1,v_2) - C(v_1,1) + C(v_1,v_2) \\ &= 1 - v_2 - v_1 + C(v_1,v_2), \end{aligned}$$

This volume, that represents a particular copula $\bar{C}$, called *survival copula* function, has marginals equal to the complement of marginals of copula C, i.e.

$$\bar{C}(1-v_1, 1-v_2) = V_C(\mathbf{S}),$$

where $\mathbf{S} = [\mathbf{v}, \mathbf{1}]$.
Computing the volume in higher dimensions can be much more complex. In cases where analytical computation of the copula is feasible, an interesting question is whether the volume can also be computed exactly, and whether such analytical computation is feasible. In Cherubini and Romagnoli (2009) a simple algorithm (CR in the following) to address this problem is provided.

Proposition (C-volume)

Given a n-dimensional copula function $C(\mathbf{u})$ with $\mathbf{u} \in [0,1]^n$, the C-volume of the box $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$ with $\mathbf{v} \in [0,1]^n$ such that $\mathbf{u} \leq \mathbf{v}$, can be represented as

$$V_C(\mathbf{S}) = \sum_{i=0}^{2^n - 1} (-1)^{k(i)} C(\mathbf{c}_n(\mathbf{p}_n(i))),$$

where $\mathbf{p}_n(i)$ denotes the n-dimensional binary representation of i, $k(i) = |\mathbf{p}_n(i)|$ counts the number of elements equal to one in $\mathbf{p}_n(i)$, and $\mathbf{c}_n(\mathbf{p}_n(i))$ is a n-dimensional vector such that $c_j = v_j$ if $p_j = 0$ and $c_j = u_j$ if $p_j = 1$, where u_j, v_j, p_j and c_j denote the jth element of the corresponding vector.

The previous representation can be specialized for *exchangeable copulas*. These are copulas where the position of the marginals can be exchanged without the aggregated result changes, i.e. $C(u, v) = C(v, u), \forall u, v$ in a 2 dimensional setting.

Proposition (exchangeable C-volume)

Given a n-dimensional exchangeable copula function $C(\mathbf{u})$ *with* $\mathbf{u} \in [0,1]^n$, *the* C*-volume of the box* $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$ *with* $\mathbf{v} \in [0,1]^n$ *such that* $\mathbf{u} \leq \mathbf{v}$, *can be represented as*

$$V_C(\mathbf{S}) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} C(\mathbf{c}_n(\bar{\mathbf{p}}_n(i))),$$

where $\bar{\mathbf{p}}_n(i)$ *denotes the n-dimensional vector with the* i*th elements equal to* 1 *and the other* $n - i$ *elements equal to* 0.

Example

To compute the survival copula of $C(\mathbf{u})$ with $\mathbf{u} \in [0,1]^3$, we consider the C-volume in the box $\mathbf{S} = [\mathbf{u}, 1]$. The algorithm consider the binary representation of all $i = 0, \ldots, 7$, representing vectors $\mathbf{p}_3(i), \forall i$, i.e.

$$\mathbf{p}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

From $\mathbf{p}_3$ we recover then matrix $\mathbf{c}_3$, substituting the coordinates of the box according to the rule explained before,

$$\mathbf{c}_3 = \begin{bmatrix} 1 & 1 & 1 \\ u_1 & 1 & 1 \\ 1 & u_2 & 1 \\ 1 & 1 & u_3 \\ u_1 & u_2 & 1 \\ u_1 & 1 & u_3 \\ 1 & u_2 & u_3 \\ u_1 & 1 & u_3 \\ u_1 & u_2 & u_3 \end{bmatrix}$$

Finally we compose copula functions evaluated for each rows of matrix $\mathbf{c}_3$ with a sign depending on the integer to which it is a transformed binary representation, i.e.

$$
\begin{aligned}
V_C(\mathbf{S}) \quad = \quad & C(1,1,1) - (C(u_1,1,1) + C(1,u_2,1) + C(1,1,u_3)) + \\
& + (C(u_1,u_2,1) + C(u_1,1,u_3) + C(1,u_2,u_3)) - C(u_1,u_2,u_3)
\end{aligned}
$$

Clusterized Hierarchical Copulas: CHY-volume

Proposition

Given a clusterized hierarchical copula function CHY with k groups of dimension $m_s, s = 1, .., k$ respectively such that $\sum_{s=1}^{k} m_s = n$, the volume of the n-dimensional box $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$ with $\mathbf{u}, \mathbf{v} \in [0,1]^n, \mathbf{u} \leq \mathbf{v}$, may be represented as:

$$V_{CHY}(\mathbf{S}) = \sum_{i=0}^{n} (-1)^i \sum_{j=1}^{D^c(i,k)} CHY(\mathbf{c}(\mathbf{p}_{i,k}(j)))$$

where the coordinates defining the box $\mathbf{S}$ are the column aggregation of matrices $\tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in [0,1]^k \times [0,1]^{\max\{m_s, s=1,\dots,k\}}$ whose (s, w)-th element corresponds to the w-th position in the s-th group

Proposition

where $D^c(i,k)$ *denotes the number of the o.c.c.ds of the integer* i *into* k *groups,* $\mathbf{p}_{i,k}(j)$ *is a* $k \times \max(m_s, s=1,...,k)$ *matrix of zeros and ones corresponding to the j-th o.c.c.d. of* i *ones into* k *groups of dimensions* $m_s, s=1,...,k$, $\mathbf{c}(\mathbf{p}_{i,k}(j))$ *is a* $k \times \max(m_s, s=1,...,k)$ *matrix such that* $c_{sw} = \tilde{v}_{sw}$ *if* $p_{sw} = 0$ *and* $c_{sw} = \tilde{u}_{sw}$ *if* $p_{sw} = 1$, *where* $\tilde{u}_{sw}, \tilde{v}_{sw}, p_{sw}$ *and* c_{sw} *denote the* (s,w)*-th element of the corresponding matrix and* $CHY(\mathbf{c}(\mathbf{p}_{i,k}(j)))$ *is the clusterized hierarchical copula computed for the j-th o.c.c.d..*

Example

We consider a 3-dimensional problem where the elements are grouped into two classes, the first with cardinality one and the second one composed by two elements. The dependence structure is represented by a clusterized hierarchical copula CHY. We compute the volume in the box $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$, where $\mathbf{u}' = [u_1, u_2, u_3]$ and $\mathbf{v}' = [v_1, v_2, v_3]$. Following the statement of the proposition, the box is represented as a matrices' product,i.e.

$$\begin{bmatrix} u_1 & u_1 \\ u_2 & u_3 \end{bmatrix} \times \begin{bmatrix} v_1 & v_1 \\ v_2 & v_3 \end{bmatrix}$$

and we generate the $\mathbf{p}_{i,2}$ matrices and the corresponding $\mathbf{c}(\mathbf{p}_{i,2}), i = 0, 1, 2, 3$ ones, i.e.

$$\mathbf{p}_{0,2}(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{0,2}(1)) = \begin{pmatrix} v_1 & v_1 \\ v_2 & v_3 \end{pmatrix}$$

$$\mathbf{p}_{1,2}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{1,2}(1)) = \begin{pmatrix} u_1 & u_1 \\ v_2 & v_3 \end{pmatrix}$$

$$\mathbf{p}_{1,2}(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{1,2}(2)) = \begin{pmatrix} v_1 & v_1 \\ u_2 & v_3 \end{pmatrix}$$

$$\mathbf{p}_{1,2}(3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{1,2}(3)) = \begin{pmatrix} v_1 & v_1 \\ v_2 & u_3 \end{pmatrix}$$

$$\mathbf{p}_{2,2}(1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{2,2}(1)) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_3 \end{pmatrix}$$

$$\mathbf{p}_{2,2}(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{2,2}(2)) = \begin{pmatrix} u_1 & v_1 \\ v_2 & u_3 \end{pmatrix}$$

$$\mathbf{p}_{2,2}(3) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{2,2}(3)) = \begin{pmatrix} v_1 & v_1 \\ u_2 & u_3 \end{pmatrix}$$

$$\mathbf{p}_{3,2}(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to \mathbf{c}(\mathbf{p}_{3,2}(1)) = \begin{pmatrix} u_1 & v_1 \\ u_2 & u_3 \end{pmatrix},$$

where we observe that the second place in the first group is fictitious since it is only useful to write the problem in matrix form.
Finally we compose the hierarchical copulas as follows,

$$\begin{aligned} V_{CHY}(\mathbf{S}) \quad = \quad & CHY(v_1, v_2, v_3) - \\ & (CHY(u_1, v_2, v_3) + CHY(v_1, u_2, v_3) + CHY(v_1, v_2, u_3)) \\ & + CHY(u_1, u_2, v_3) + CHY(u_1, v_2, u_3) \\ & + CHY(v_1, u_2, u_3) - CHY(\underline{u_1, u_2, u_3}). \end{aligned}$$

We can observe that the computation of the Survival copula function is a particular case of the volume computation;

Corollary

(Survival copula function in n dimension) The survival of a CHY with marginals $\mathbf{u} \in [0,1]^k$, *may be represented as:*

$$V_{CHY}(\mathbf{S}) = \sum_{i=0}^{n}(-1)^i \sum_{j=1}^{\hat{D}^c(i,k)} CHY(\mathbf{c}(\mathbf{p}_{i,k}(j)))$$

where the box $\mathbf{S} = [\mathbf{u}, \mathbf{e}]$ *and where* $\mathbf{e} = (1, 1, ..., 1)$ *is the unit vector in* $\mathbb{R}^k$.

Table of Contents

Origin and structure of CAT bonds

- Originated in the mid-1990s to protect the insurance sector from catastrophe risk;
- Purpose: transfer catastrophe risk from originators to the market;
- Benefits: risk transfer and yield-enhancement;

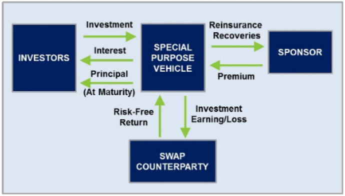


Figure: Scheme of a typical CAT bonds transaction

Table of Contents

Natural shocks

CATs are affected by a particular type of shock - called "natural shock" and defined as follow:

1. natural shocks represent occurrence of extreme natural events, such as hurricanes, heat waves, etc.;
2. natural shocks occurrence is increasing with climate change;
3. natural shocks trigger CAT bonds' default conditions;
4. it is not possible to assess with certainty the probability of occurrence;
5. natural shocks are correlated with financial shocks;

Condition (4) introduces a subjective perception in the risk's measure that's referred as "ambiguity".
His effect is here a mispricing on CATs - hence the premiums are not consistent with the underlying risk.

Including CAT bonds

Assumption

We assume that bank i is the issuer of the ith CAT related to the ith natural event. She buys an insurance (through the CAT) from every other actors (paying a total premium $\hat{CAT}_i^P$, related to $e_i(1)$), and sells insurance (through CATs) written on the natural events $j \neq i$ to every other actor (receiving a total premium $\hat{CAT}_i^R$, related to $e_i(1)$). These products give protection to the issuers for the jth event because in case of its happening the buyers of CAT (i.e. the seller of the insurance) will have to repay the total loss to the issuer, i.e. $\sum_j L_j(\chi_i^{j,CAT})$, where also the losses are related to $e_i(1)$).

CAT bonds are related to natural shocks, that we denote as $\hat{u}$. We can hence specify the default condition of bank i as $f(u_i, \hat{u}_i) < 0$ or in term of u_i:

$$u_i < \theta_i^I + \theta_i^{II}$$

where

$$\theta_i^I = \frac{-\varepsilon_i \mu_i - 1}{\sigma_i \varepsilon_i}$$

$$\theta_i^{II} = \frac{\sum_j L_j(\chi_i^{j,CAT}) - \Delta C\hat{A}T_i}{\sigma_i \varepsilon_i}$$

where we denote as $\chi_i^{j,CAT}$ the default indicator for the catastrophe bonds bought by bank i and $\Delta C\hat{A}T_i = C\hat{A}T_i^R - C\hat{A}T_i^P$.
Hence the default condition is a combination of two parts: one related to "standard" bank assets and one related to assets depending on natural events.

Table of Contents

Shocks and Natural events

The rule that governs the joint arrival of financial and natural shocks is defined with an *Arrival matrix (see Bernardi and Romagnoli, 2016):*

Definition: Arrival matrix

Given a set of $N = n + \hat{n}$ *random variables* $\{u_s, s = 1, ..., n; \hat{u}_{\hat{s}}, \hat{s} = 1, ..., \hat{n}\}$ *where u define the size of the shock while* $\hat{u}$ *identify the happening of the natural events, whose marginal probabilities, i.e. the probabilities that the event happens, are* $\{p_{\hat{s}}, \hat{s} = 1, ..., \hat{n}\}$, *the arrival matrix is a stochastic matrix* **A**, *whose components* a_{ji} *are dependent r.vs such that*

$$i = 1, ..., n \quad a_{ji} \sim \mathcal{N}(0, v_j)$$

$$i = n+1, ..., n+\hat{n} \quad a_{ji} = \begin{cases} 1 & \textit{with probability } p_i \\ 0 & \textit{with probability } 1 - p_i \end{cases}$$

and where the dependency is induced by the copula $C_{u,\hat{u}}$.

Mispricing and garbling density

To specify the mispricing on CAT bonds, we use a garbling density. Based on the true margins $\{p_{\hat{s}}, \hat{s} = 1, ..., \hat{n}\}$, for the Cat bought by the ith actor and related to the jth natural event, issued by the jth actor we have:

$$cat_i^j = \int_{\mathcal{S}^j} L^j(\chi_i^{j,CAT})dF^j$$

But this is not observable (see Klibanoff et al., 2005), hence we write:

$$mcat_i^j = \int_{\mathcal{O}^j} \int_{\mathcal{S}_j} L^j(\chi_i^{j,CAT})d\pi(L^j)d\mu(\pi)$$

where π is the given prior information of the jth actor.

Mispricing and garbling density

The dependence structure between π and μ is represented in term of a garbling density which gives a distortion to the true joint probability. We note the copula function between π and μ as $C_{\pi,\mu}(o^j, s^j) = \underbrace{\mu(o^j|s^j)}\pi(s^j), o^j \in \mathcal{O}^j, s^j \in \mathcal{S}^j$, where $\mu(o^j|s^j)$ stands for the garbling density of the jth actor.
Hence if $C_{\pi,\mu}(o^j, s^j) = \mu(o^j|s^j)\pi(s^j)$, we have the following CAT premium:

$$mcat_i^j = \int_{\mathcal{O}^j \times \mathcal{S}^j} L^j(\chi_i^{j,CAT}) dC_{\pi,\mu}(o^j, s^j)$$

and $\hat{CAT}_i^R = \sum_{j,j\neq i} \frac{mcat_i^j}{e_i(1)}$, $\hat{CAT}_i^P = \sum_{j,j\neq i} \frac{mcat_j^i}{e_i(1)}$.

Mispricing and garbling matrix in action

We consider a discrete example where the r.v. X has a range determined by the state of the world ω and detailed in 3 states, i.e. low (≤ 90), mid (in $(90, 100]$) and high ($>$100). The DM is informed about the states through the signals that he receives; the number of signals and their quality determine the level of information of the DM (sees Blackwell, 1953).
Under full information assumption, the prior is represented by a 3×3 identity matrix $\mathbf{I}$, whose rows are the states and whose columns are the signals. These signals are then received by the DM who is ambiguous on them; his ambiguity behavior is given by the garbling matrix, i.e.

$$\mathbf{G} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.3 & 0.7 \end{pmatrix}$$

Mispricing and garbling matrix in action

The product of the information matrix and the garbling one, $\mathbf{I} \times \mathbf{G}$ determines a distortion on the range of the r.v. and then causes a mispricing. In case of full information $\mathbf{I} \times \mathbf{G} = \mathbf{G}$ and if we assume a punctual position of the signal in 85, 95 and 105 respectively, given a true value of 85 (low state), his garbled value will be:

$$85 \times 0.5 + 95 \times 0.5 = 90$$

Imperfect or Partial information are represented by matrices as:

$$\mathbf{II} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{IP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ambiguity adversion/love

A LT garbling matrix, that assigns weights to the lowest states, represents an ambiguity lover DM, while a UT garbling matrix characterizes an ambiguity adverse DM. A Symmetric garbling acts as a factor of dispersion around the true value with equal weights on the lowest and highest states; it doesn't means a particular attitude (aversion or love) with respect to the ambiguity nevertheless it introduces vaguenesses into the model.

$$\mathbf{LT} = \begin{pmatrix} 1 & 0 & 0 \\ 0.7 & 0.3 & 0 \\ 0.35 & 0.35 & 0.3 \end{pmatrix} \qquad \mathbf{UT} = \begin{pmatrix} 0.3 & 0.35 & 0.35 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{SY} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

How the garbling works

Given the signal $[0.05; 0.02; 0.06; 0.07; 0.01]$ representing the true probabilities of the cat events and the garbling of the issuers, i.e.

$$\mathbf{UT} = \begin{pmatrix} 0.3 & 0.35 & 0.35 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

we partition the range of probabilities in 3 different interval, i.e. $p_L([0, 0.05)), p_M([0.05, 0.09)), p_H(>= 0.09)$ and assign a representative value to these classes, for example $[0.01; 0.07; 0.09]$. Hence the signal belongs in a class which classify the row of the garbling, i.e.

$$if \quad s_i \in p_L, mcat_i = UT(1, :) * [0.01; 0.07; 0.09]$$

Here is $mcat = [0.059; 0.059; 0.076; 0.076; 0.059]$.

Table of Contents

External assets and CAT: default in term of the natural shocks

We can define the default condition at time 2 of actor i when its equity at time 2 is negative, i.e. $a_i(2) - l_i(2) < 0$. In equity at time 1 relative terms, the condition becomes:

$$\varepsilon_i(1 + \mu_i + \sigma_i u_i) + \xi_i \Delta \hat{CAT}_i - \sum_j L_j(\chi_i^{j,CAT}) - (\varepsilon_i - 1) < 0$$

$$\sum_j L_j(\chi_i^{j,CAT}) > 1 + \xi_i \Delta \hat{CAT}_i + \varepsilon_i(\mu_i + \sigma_i u_i)$$

$$\sum_j \chi_i^{j,CAT} > \underbrace{\frac{1 + \xi_i \Delta \hat{CAT}_i}{\xi_i}} + \underbrace{\frac{\varepsilon_i(\mu_i + \sigma_i u_i)}{\xi_i}}$$

where ξ_i is the leverage on CAT of actor i.

External assets and CAT: default in term of the natural shocks

Hence the individual default probability is defined as:

$$PD^i = \mathbb{P}\left(\sum_j \chi_i^{j,CAT} > T_1 + T_2\right) = 1 - \mathbb{P}\left(\sum_j \chi_i^{j,CAT} \leq T_1 + T_2\right)$$

where $T_1 = \frac{1+\xi_i \Delta \hat{CAT}_i}{\xi_i}$ and $T_2 = \frac{\varepsilon_i(\mu_i+\sigma_i u_i)}{\xi_i}$.

Aggregation Problem

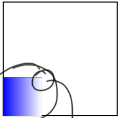
Given a set of random variables, whose dependence structure is known, we are interested in counting the happening of a random event. If we focus on the happening of the catastrophic events which may affect the network, our aim is to recover the *loss distribution* of every actor.

Aggregation algorithm in 2-dimension: the idea

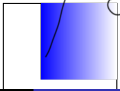
All possible scenarios are given by the c.d. of the catastrophic events (see Beranardi and Romagnoli, 2016), i.e.

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{array}{l} \rightarrow 0 \text{ cat} \\ \rightarrow 1 \text{ cat} \\ \rightarrow 1 \text{ cat} \\ \rightarrow 2 \text{ cats} \end{array}$$

- The event $(1,1)$ has a probability to happen $V_C((0,0) \times (p_X, p_Y))$

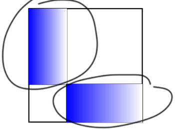


- The event $(0,0)$ has a probability to happen $V_C((p_X, p_Y) \times (1,1))$



Aggregation algorithm in 2-dimension: the idea

- The events $(1, 0)$ and $(0, 1)$ have probabilities $V_C((0, p_Y) \times (p_X, 1))$ and $V_C((p_X, 0) \times (1, p_Y))$, respectively.



2-dimensional Loss function

The loss function is given by:

$$
\begin{array}{ccc}
0 & \rightarrow & V_C((p_X, p_Y) \times (1, 1)) \\
1 & \rightarrow & V_C((0, p_Y) \times (p_X, 1)) + V_C((p_X, 0) \times (1, p_Y)) \\
2 & \rightarrow & V_C((0, 0) \times (p_X, p_Y))
\end{array}
$$

Aggregation algorithm in 2-dimension: the idea

We observe that:

$$\begin{aligned}
V_C((0,0)\times(p_X,p_Y)) &= C_{XY}(p_X,p_Y)\\
V_C((p_X,p_Y)\times(1,1)) &= p_X+p_Y-1+C_{XY}(1-p_X,1-p_Y)\\
V_C((0,p_Y)\times(p_X,1)) &= p_X - C_{XY}(p_X,p_Y)\\
V_C((p_X,0)\times(1,p_Y)) &= p_Y - C_{XY}(p_X,p_Y)
\end{aligned}$$

C_{XY} is always increasing in the correlation parameter (linear correlation and Kendal tau), hence $V_C((0,0)\times(p_X,p_Y))$ and $V_C((p_X,p_Y)\times(1,1))$ are also increasing while $V_C((0,p_Y)\times(p_X,1))$ and $V_C((p_X,0)\times(1,p_Y))$ are decreasing.

2-dimensional complete network: individual PD

Example: $p_X = 0.3, p_Y = 0.1$, Gaussian Copula, results in $\mathbb{P}(X+Y>0), \mathbb{P}(X+Y \leq 1)$ decreasing in the correlation while $\mathbb{P}(X+Y>1) = \mathbb{P}(X+Y=2)$ increasing in the correlation.

2-dimensional complete network without external asset: individual PD

If $\xi_i \leq 1, \Delta\hat{CAT}_i \geq 0$ the individual default probability is given by:

$$PD^i = \mathbb{P}\left(\sum_j \chi_i^{j,CAT} > T_1\right) = \mathbb{P}\left(\sum_j \chi_i^{j,CAT} = 2\right)$$

where $T_1 = \frac{1+\xi_i \Delta\hat{CAT}_i}{\xi_i} > 1$. It is increasing in the correlation.

If $\xi_i > 1$, T_1 will decrease and the default may occur also for $\sum_j \chi_i^{j,CAT} < 2$ and hence the default probability could be decreasing in the correlation.

5-dimensional complete network, 2 extreme events: individual default probability

We assume to have actors 1,2 and 3 who issue a CAT bond written on event A and actors 4 and 5 who issue a CAT on event B. The possible scenarios are:

$$
\begin{array}{ccllll}
eventA & eventB & & & & \\
0 & 0 & \rightarrow concordant & scenario = 0 & events & \\
1 & 0 & \rightarrow discordant & scenario = 3 & A, 0 & B \\
0 & 1 & \rightarrow discordant & scenario = 0 & A, 2 & B \\
1 & 1 & \rightarrow concordant & scenario = 5 & events &
\end{array}
$$

The concord scenarios lead to a probability increasing in the correlation while the discordant ones are decreasing on the correlation. Hence if the threshold $T_1 > 2$ the actor-1 default probability is increasing in the correlation. If $T_1 \geq 2$ the increasingness with the correlation is not assured.

Aggregation algorithm in n-dimension: the idea

DC volume: Given a copula function of dimension n, and an arrival matrix $\mathbf{A} = \{\mathbf{p}_{k,n}, k \in \mathbb{N}, k \leq n\}$, the volume of the DC defined by the n-dimensional box $\mathbf{S} = [\mathbf{u}, \mathbf{v}]$ with $\mathbf{u}, \mathbf{v} \in [0,1]^n, \mathbf{u} \leq \mathbf{v}$, may be represented as:

$$V_{DC}(\mathbf{S}) = \sum_{k=0}^{n} (-1)^k \sum_{j=1}^{D^d(k,n)} DC(\mathbf{c}(\mathbf{p}_{k,n}(j)))$$

where $D^d(k,n)$ denotes the number of distorted combinatorial distribution of the integer k into n places, $\mathbf{p}_{k,n}(j)$ is the j-th rows of the sub-arrival matrix $\mathbf{p}_{k,n}$ whose dimension is $D^d(k,n) \times n$, $\mathbf{c}(\mathbf{p}_{k,n}(j))$ is a vector of dimension n such that $c_{j,i} = v_i$ if $p_{j,i} = 0$ and $c_{j,i} = u_i$ if $p_{j,i} = 1$, where u_i, v_i are the i-th element of the corresponding vector and $c_{j,i}$ denotes the (j,i)-th element of the corresponding vector and $DC(\mathbf{c}(\mathbf{p}_{k,n}(j)))$ is the copula computed for the j-th d.c.d. of the sub-arrival matrix $\mathbf{p}_{k,n}$.

Aggregation algorithm in n-dimension: the idea

We define the negative and the positive part of $V_{DC}(\mathbf{S})$:

$$V_{DC}(\mathbf{S})^- = \sum_{k \in \mathbb{N}^o, k=1}^{n} (-1)^k \sum_{j=1}^{D^d(k,n)} DC(\mathbf{c}(\mathbf{p}_{k,n}(j)))$$

$$V_{DC}(\mathbf{S})^+ = \sum_{k \in \mathbb{N}^e, k=0}^{n} (-1)^k \sum_{j=1}^{D^d(k,n)} DC(\mathbf{c}(\mathbf{p}_{k,n}(j)))$$

The concordant scenarios have a probability always increasing in the correlation. On the other hand the discordant cases have a probability decomposable in a positive part, that is increasing in the correlation, and a negative one, that is decreasing. The dominant effect cannot be clearly determined.

Table of Contents

Fuzzy copula models

$(\Omega, \mathcal{F}, \mu)$ $\quad \mu(A \cup B) < \mu(A) + \mu(B) \rightarrow$ sub-additive

$\mu(A \cup B) > \mu(A) + \mu(B) \rightarrow$ super-additive

$A, B \in \Omega$

We have two different ways to introduce a further level of ambiguity on the copula function involved in the pricing:

- make the marginals ambiguous
- make the dependence structure ambiguous

In this setting the mispricing results into an interval of prices depending on a g_λ-measure as the Sugeno one and its dual super-additive measure, which define the span of the core space.

$$\mathcal{C} = \{ \xi : \underline{\mu}(A) \leq \xi(A) \leq \bar{\mu}(A),\ \underline{\mu}(A) = 1 - \bar{\mu}(A),\ \forall A \in \mathcal{F} \}$$

Conditional Fuzzy copulas

Given the incompleteness of such a market, the pricing kernel cannot be unique but we have a set of kernels $\mathcal{S}_\lambda^F$ that, for a set of measurable functions $\gamma_i, \forall i$ and for the invariance property of copulas for conditional fuzzification, is given by the following

$\mathcal{S}_\lambda^F =$ Choquet

$$\left\{ S\left((C)\int \gamma_1(u_1)d\xi_1, \ldots, (C)\int \gamma_n(u_n)d\xi_n\right), \xi_i \in \mathcal{C}_i, \forall i; S^L \leq S \leq S^U \right\},$$

where $u_i = \mathbb{P}(X_i \leq K_i)$ and $\gamma_i(u_i)$ is the cf of $u_i, \forall i$,

$\mathcal{C}_i = \{\xi_i; g_\lambda^{F_i}(u_i) \leq \xi_i \leq g_{\lambda^*}^{F_i}(u_i)\}$ and where

$$S^L = S\left((C)\int \gamma_1(u_1)dg_\lambda^{F_1}, \ldots, (C)\int \gamma_n(u_n)dg_\lambda^{F_n}\right) \text{ and}$$

$$S^U = S\left((C)\int \gamma_1(u_1)dg_{\lambda^*}^{F_1}, \ldots, (C)\int \gamma_n(u_n)dg_{\lambda^*}^{F_n}\right).$$

Unconditional Fuzzy copulas

On the other hand a fuzzified version of the top-down approach would be focused on fuzzified multivariate process whose dependency structure is attached by ambiguity as its arguments themselves. To this aim we define the *fuzzy kernel* that is given by

$$\mathcal{CS}_\lambda^F = \left\{ CS_\lambda^F = (C)\int S(u_1,\ldots,u_n)\, d\bigcirc_{i=1}^n \xi_i;\ \underline{CS}_\lambda^F \leq CS_\lambda^F \leq \overline{CS}_\lambda^F, \lambda \in \mathbb{R} \right\}$$

where $\underline{CS}_\lambda^F = (C)\int S(u_1,\ldots,u_n)\, d\bigcirc_{i=1}^n g_\lambda^{F_i}$ and $\overline{CS}_\lambda^F = (C)\int \overline{S}(u_1,\ldots,u_n)\, d\bigcirc_{i=1}^n g_{\lambda^*}^{F_i}$, where $g_\lambda^{F_i}$ stands for the g_λ-measure corresponding to the ith coordinate of F.

Table of Contents

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Thank you very much!

For any question, feel free to ask or contact
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